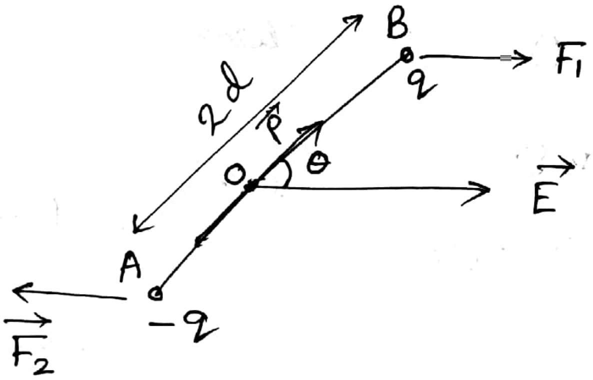
Forces and Torques on dipoles :-



A dipole AB is placed in an uniform electric field $\vec{E}$.

O is the mid point of AB and $AB = 2d$.

Let at any instant dipole axis is making an angle "θ" with the electric field.

Torque acting on the dipole,

$$\vec{\tau} = \vec{\tau_1} + \vec{\tau_2} = q(\vec{OB} \times \vec{E}) + q(\vec{AO} \times \vec{E})$$

$$= q(\vec{OB} + \vec{AO}) \times \vec{E}$$

$$= q(\vec{AB}) \times \vec{E}$$

$$= q(2d) \times \vec{E}$$

$$\vec{\tau} = \vec{P} \times \vec{E} = PE\sin\theta$$

Workdone in rotating this dipole through an angle "$d\theta$" in this field, $dW = \tau\, d\theta$

Total work done in rotating the dipole ~~in~~ from an angle "θ_1" to θ_2,

$$W = \int_{\theta_1}^{\theta_2} \tau\, d\theta = \int_{\theta_1}^{\theta_2} PE\sin\theta\, d\theta = \cancel{PE(1-\cos\theta)}$$

$$\omega = -PE\,(\cos\theta_2 - \cos\theta_1)$$

If $\theta_1 = 90^\circ$, $\theta_2 = \theta$, then

$$\omega = -PE\cos\theta$$

Therefore in this case, ~~change of~~ the Potential ener or workdone,

$$U = \omega = -PE\cos\theta = -\vec{P}\cdot\vec{E}$$

Again force acting on the poles of the dipole,

$$F = qE$$
$$= q(2d)\,\nabla E$$
$$= (P\cdot\nabla)E$$
$$= \nabla(\vec{P}\cdot\vec{E})$$

∇E

$\Rightarrow \frac{\partial E}{\partial(2d)}$

$\frac{\partial}{\partial x}$

$2d$

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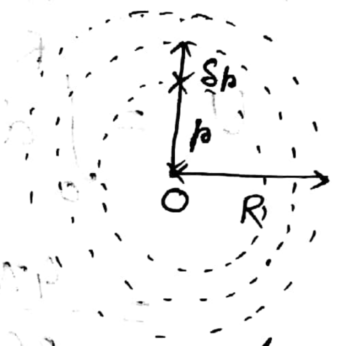
Electrostatic Energy of a charged sphere :-

Let us consider a sphere of radius R and charge density ρ.
Let us imagine that the sphere is assembled by building their spherical layers of infinitesimal thickness in succession.



If q_p is the charge of the sphere, when it has been built up to the radius "p".

Work done in bringing a further charge "δq" from infinity to the layer of thickness "δp" is

$$\Delta W = \Delta U = \frac{q_p \, \delta q}{4\pi\epsilon_0 p}$$

If ρ is the charge density, then charge in the sphere of radius p is given by,

$$q_p = \frac{4}{3}\pi p^3 \rho$$

The charge in the shell of thickness δp,

$$\delta q = 4\pi p^2 \delta p \, \rho = 4\pi p^2 \rho \, \delta p$$

$$\therefore \quad \Delta U = \frac{q_p}{4\pi\epsilon_0 p} 4\pi p^2 \rho \, \delta p$$

$$= \frac{\frac{4}{3}\pi p^3 \rho}{4\pi\epsilon_0 p} 4\pi p^2 \rho \, \delta p$$

$$\Delta U = \frac{4\pi}{3\epsilon_0} \rho^2 p^4 \delta p$$

Total energy required to assemble the sphere of radius R is

$$U_i = \int_0^R du = \frac{4\pi}{3\epsilon_0}\rho^2 \int_0^R r^4 dr$$

$$= \frac{4\pi}{3\epsilon_0}\rho^2 \times \frac{R^5}{5}$$

$$= \frac{4\pi\rho^2 R^5}{15\epsilon_0}$$

$$\left(\frac{4}{3}\pi R^3 \rho = q\right)$$

$$= \frac{16\pi^2 \times 3 \times R^6 \times \rho^2}{4\pi\epsilon_0 \times 3^2 \times 5 \times R}$$

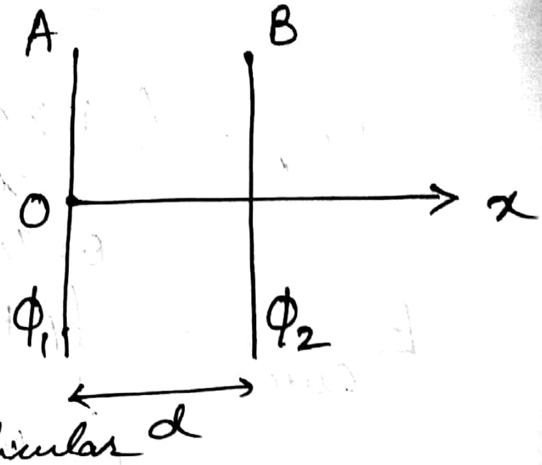
$$\Rightarrow q^2 = \frac{16\pi^2}{9} R^6 \rho^2$$

$$= \frac{16\pi^2}{9} R^6 \rho^2 \frac{3}{5 \times 4\pi\epsilon_0 R}$$

$$\boxed{U = \frac{3}{5}\frac{q^2}{4\pi\epsilon_0 R}}$$

Solution of Laplace equations

(1) Potential and field at any point between the plates of a parallel plate condenser (capacitor)



Ans: Let ϕ_1 and ϕ_2 be the potentials of the plates A and B. Here $\phi_1 > \phi_2$.

Let the plates A and B of the capacitor being perpendicular to the x-axis. "O" is the origin. The potential ϕ at any point between the plates will depend on x only and the corresponding form of Laplace's equation is,

$$\frac{\partial^2 \phi}{\partial x^2} = 0 \Rightarrow$$

on integration, $$\frac{\partial \phi}{\partial x} = C_1$$

$$\Rightarrow \phi = C_1 x + C_2$$

C_1 and C_2 are constants, to be determined from boundary condition,

when $x = 0,\ \phi = \phi_1$

$x = d,\ \phi = \phi_2$

$$\therefore \phi_1 = C_2$$

$$\phi_2 = C_1 d + C_2 = C_1 d + \phi_1$$

$$\Rightarrow C_1 = \frac{\phi_2 - \phi_1}{d}$$

$$\therefore \phi = \left(\frac{\phi_2 - \phi_1}{d}\right) x + C_2$$

$$\phi = -\left(\frac{\phi_1 - \phi_2}{d}\right) x + \phi_1$$

This gives the potential at any point "x" between the plates of a parallel plate capacitor. The surface density "σ" on the plate A can be as,

$$E = -\frac{d\phi}{dx} = \frac{\phi_1 - \phi_2}{d}$$

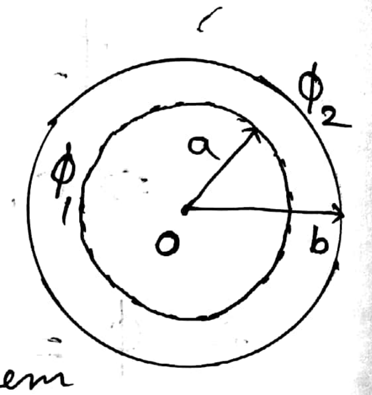
$$\Rightarrow \frac{\sigma}{\epsilon_0} = \frac{\phi_1 - \phi_2}{d} \quad \text{as } E = \frac{\sigma}{\epsilon_0}$$

$$\Rightarrow \frac{\sigma A}{\phi_1 - \phi_2} = \frac{\epsilon_0 A}{d} \Rightarrow \boxed{\frac{Q}{\phi_1 - \phi_2} = \frac{\epsilon_0 A}{d}}$$

$$\therefore C = \frac{Q}{\phi} = \frac{\epsilon_0 A}{d}$$

(b) Potential due to a Spherical capacitor

Let us consider two conducting concentric spheres of radii "a" and b. b > a, which have potentials ϕ_1 and ϕ_2 respectively.



The potential between them would be a function of ϕ and will depend on r only. The corresponding Laplace's equation in Spherical polar co-ordinates is,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = 0$$

On integration,

$$r^2\frac{\partial\phi}{\partial r} = c_1$$

$$\Rightarrow \frac{\partial\phi}{\partial r} = \frac{c_1}{r^2}$$

$$\Rightarrow \phi = -\frac{c_1}{r} + c_2$$

Here c_1 and c_2 are arbitrary constants.

When $r = a$, $\phi = \phi_1$

$r = b$ $\phi = \phi_2$

$$\therefore \phi_1 = -\frac{c_1}{a} + c_2$$

$$\phi_2 = -\frac{c_1}{b} + c_2$$

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$$\therefore \quad c_1 = -\frac{ab(\phi_1 - \phi_2)}{b-a}$$

$$c_2 = \frac{b\phi_2 - a\phi_1}{b-a}$$

$$\therefore \quad \phi = \frac{ab(\phi_1 - \phi_2)}{(b-a)\,p} + \frac{b\phi_2 - a\phi_1}{b-a}$$

This is the required result.

Again,

$$E = -\frac{\partial \phi}{\partial p}, \quad E = \frac{\sigma}{\epsilon_0}$$

$$\Rightarrow \quad \frac{\sigma}{\epsilon_0} = -\frac{c_1}{p^2} = \frac{ab(\phi_1 - \phi_2)}{(b-a)\,p^2}$$

Surface density σ on the sphere $p = a$ is given by

$$\sigma = \frac{\epsilon_0}{a^2}\,\frac{ab(\phi_1 - \phi_2)}{(b-a)}$$

or

$$Q = 4\pi a^2 \sigma = 4\pi \epsilon_0 \frac{ab(\phi_1 - \phi_2)}{(b-a)}$$

$$\therefore \quad c = \frac{Q}{(\phi_1 - \phi_2)} = 4\pi \epsilon_0 \frac{ab}{(b-a)}$$

(i) When $b \to \infty$,

$$c = 4\pi\epsilon_0 \frac{a}{1 - a/b} \simeq 4\pi\epsilon_0 a$$

$$\phi = \frac{Q}{c} = \frac{Q}{4\pi\epsilon_0 a}$$

Which are capacity and potential of sphere having radius "a".